Sharp large deviation principles for descents of random permutations

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Plan of the talk

- Random permutations and descents
- Large Deviation Principles.
- Descent statistics.
- Descent statistics for the couple.
- Construction of the probability space.

Random Permutations

We denote by π_n a **random uniform permutation** in S_n the **symmetric group** on $\{1, 2, \ldots, n\}$.

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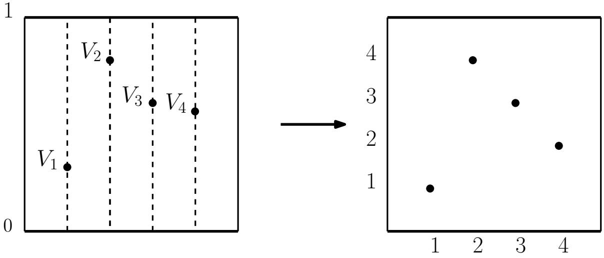

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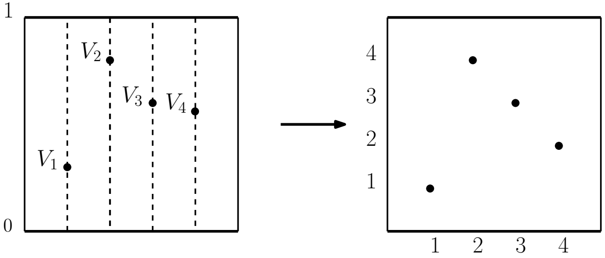

The (total) number of **descents** in a permutation π_n is given by

$$D_n = D_n(\pi_n) = \sum_{k=1}^{n-1} \mathbb{1}_{\{\pi_n(k+1) < \pi_n(k)\}}$$

Theorem: (Tanny '73)

D_n is equal in distribution to $\lfloor S_n \rfloor$, for S_n defined as

$$S_n = \sum_{k=1}^{n-1} U_k,$$

where $U_k = \mathrm{Unif}[0,1]$ independent of the others.

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Sketch of proof

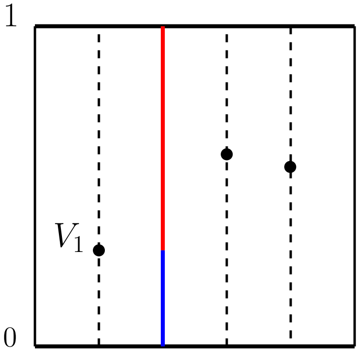
Let $\mathcal{U} = (U_1, U_2, \ldots, U_n)$ I.I.D. Unif$[0,1]$ r.v.
We define $\mathcal{V} = (V_1, V_2, \ldots, V_n)$ as

$$V_i = \{U_1 + U_2 + \cdots + U_i\} \qquad \forall i \in \{1, 2, \ldots, n\},$$

where $\{x\} = x - \lfloor x \rfloor$, i.e. the fractional part.

Proposition

The collection $\mathcal{V}$ is an i.i.d. collection of Unif$[0,1]$ *r.v.*



- V_2 will create an ascent = red part = $(\lfloor U_1 + U_2 \rfloor - \lfloor U_1 \rfloor = 0)$
- V_2 will create a descent = blue part = $(\lfloor U_1 + U_2 \rfloor - \lfloor U_1 \rfloor = 1)$

The random variable S_n follows the Irwin-Hall (continuous) distribution.

Corollary

$$\mathbb{E}(D_n) = \frac{n-1}{2} \quad \textit{and} \quad \mathbb{V}ar(D_n) = \frac{n+1}{12} \qquad \forall n \geq 2$$

Theorem : LLN (D. Freedman '65)

$$\lim_{n\to\infty} \frac{D_n}{n} = \frac{1}{2} \quad a.s.$$

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$$\mathbb{P}\left(D_n/n > x\right) = \mathbb{P}(\lfloor S_n \rfloor > nx) = \mathbb{P}(S_n > \lceil nx \rceil)$$

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Theorem : CLT (David & Barton '62, Harper '67, Bender '73, +...)

$$\sqrt{n}\left(\frac{D_n}{n} - \frac{1}{2}\right) \xrightarrow{(d)} \mathcal{N}\left(0, \frac{1}{12}\right)$$

Large deviation principles (LDP)

Main idea : give an adjusted behavior of the probability off the mean.

Let $D_n \in L_1$ with mean m, $x > m$ and $t \geq 0$.

$$\begin{aligned}
\mathbb{P}\left(\frac{D_n}{n} > x\right) &= \mathbb{P}(\exp(tD_n) > e^{tnx}) \\
&\leq \exp(-tnx)\underbrace{\mathbb{E}(\exp(tD_n))}_{L_n(t)} \\
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$$\mathbb{P}\left(\frac{D_n}{n} > x\right) \leq \exp\left(-nI(x)\right)$$

The idea is to prove that this is **"tight"** as inequality.

Rough LDP (RLDP) : Determine the power in the optimal exponential speed

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}\left(\frac{D_n}{n} > x\right)\right) = -I(x)$$

Sharp LDP (SLDP) : Determine a more adjusted asymptotic behavior, i.e.

$$\mathbb{P}\left(\frac{D_n}{n} > x\right) \approx c(x,n)\exp\left(-nI(x)\right) \quad \text{as } n \to \infty$$

Theorem : RLDP (Bryc, Minda and Sethuraman '09)

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\mathbb{P} \left(\frac{D_n}{n} > x \right) \right) = -I(x) \qquad \forall x \in (1/2, 1),$$

where the log-Laplace transform is given by

$$L(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\underbrace{\mathbb{E} \left(\exp \left(t D_n \right) \right)}_{L_n(t)} \right) = \log \left(\frac{\exp(t) - 1}{t} \right)$$

Recently Bercu, Bonnefont and Richou (BBR) obtained for an specific probability space that

$$D_{n+1} - D_n = \xi_{n+1},$$

where the conditional distribution of ξ_{n+1} given $\mathcal{F}_n = \sigma(D_1, D_2, \ldots, D_n)$ follows a $\mathcal{B}(p_n)$ with

$$p_n = \frac{n - D_n}{n+1}.$$

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Then they proved that the following is a Martingale with respect to $\mathcal{F}_n$

$$M_n = n\left(D_n - \frac{n-1}{2}\right).$$

Consequences

They derived a LLG, together with :

Theorem : Quadratic Strong Law (BBR '23)

$$\lim_{n\to\infty} \frac{1}{\log(n)} \sum_{k=1}^{n} \left(\frac{D_k}{k} - \frac{1}{2} \right)^2 = \frac{1}{12} \qquad a.s.$$

Theorem : Law of Iterated Logarithm (BBR '23)

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log(n)} \right)^{1/2} \left| \frac{D_n}{n} - \frac{1}{2} \right| = \frac{1}{\sqrt{12}} \qquad a.s.$$

Theorem : Functional CLT (BBR '23)

$$\sqrt{n} \left(\frac{D_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \frac{1}{2}, t \geq 0 \right) \xrightarrow{(d)} (W_t : t \geq 0)$$

where $(W_t : t \geq 0)$ is a real-valued centered Gaussian process starting at the origin with $\mathbb{E}(W_s W_t) = s/(12t^2)$ for all $0 < s \leq t$.

Theorem : SLDP (BBR '23)

For every $x \in (1/2, 1)$ one has

$$\mathbb{P}\left(\frac{D_n}{n} > x\right) = \frac{\exp(-nI(x) - \{nx\}t_x)}{\sigma_x t_x \sqrt{2\pi n}} (1 + o(1)),$$

where t_x is the unique solution of $L'(t_x) = x$ and $\sigma_x^2 = L''(t_x)$.

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where t_x is the unique solution of $L'(t_x) = x$ and $\sigma_x^2 = L''(t_x)$.

The theorem also holds for $y = 1 - x$ in $(0, 1/2)$, since by a symmetry argument

$$\mathbb{P}\left(\frac{D_n + 1}{n} < y\right) = \mathbb{P}\left(\frac{D_n}{n} > x\right)$$

Collaboration

Recall that π_n is a uniform r.v. in S_n. We define the descents of the inverse as

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Recall that π_n is a uniform r.v. in S_n. We define the descents of the inverse as

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What about the behavior of (D_n, D_n')?

$$Cov(D_n, D_n') = (n-1)/2n$$

Theorem : Joint CLT(Vatutin '96)

$$\sqrt{n}\left(\frac{D_n}{n} - \frac{1}{2}, \frac{D_n'}{n} - \frac{1}{2}\right) \xrightarrow{(d)} \mathcal{N}\left(\vec{0}, \frac{1}{12}\mathrm{Id}_2\right)$$

We construct a probability space which let us obtain.

Consequences

Theorem : Quadratic Strong Law (BBFR '24)

$$\lim_{n\to\infty} \frac{1}{\log(n)} \sum_{k=1}^{n} \left\langle \left(\frac{D_k}{k} - \frac{1}{2}, \frac{D'_k}{k} - \frac{1}{2} \right), u \right\rangle^2 = \frac{\|u\|^2}{12} \qquad a.s.$$

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For every $x, y \in (1/2, 1)$ it holds that

$$\mathbb{P}\left(\frac{D_n}{n-1} > x, \frac{D'_n}{n-1} > y\right) = \frac{\exp(-n(I(x)+I(y)) - \{nx\}t_x - \{ny\}t_y + t_x t_y/2)}{\sigma_x t_x \sigma_y t_y 2\pi n}(1+o(1)),$$

where t_x is the unique solution of $L'(t_x) = x$ and $\sigma_x^2 = L''(t_x)$.

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Corollary

The sequence $(D_n/n, D'_n/n)$ satisfies an RLDP with rate function given by $I(x,y) = I(x) + I(y)$.

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For every $x \in (1/2, 1)$ one has

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Theorem : SLDP (BBFR '24)

For every $x \in (1/2, 1)$ and any $p \geq 1$ there exists a sequence $d_{n,1}(x), \ldots, d_{n,p}(x)$ such that for n large enough one has

$$\mathbb{P}\left(\frac{D_n}{n} > x\right) = \frac{\exp(-nI(x) - \{nx\}t_x)}{\sigma_x t_x \sqrt{2\pi n}} \left(1 + \sum_{k=1}^{p} \frac{d_{n,k}(x)}{n^k} + o\left(\frac{1}{n^{p+1}}\right)\right),$$

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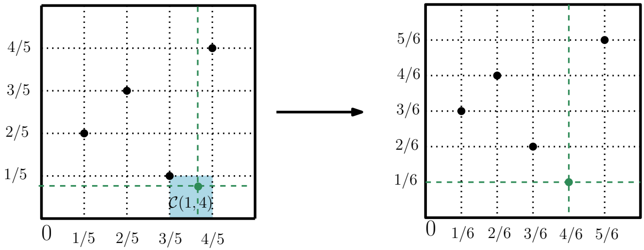
For example

$$d_{n,1}(x) = \frac{1}{\sigma_x^2}\left(\frac{\ell_4}{8\sigma_x^2} - \frac{5\ell_3^2}{24\sigma_x^4} - \frac{\ell_3}{2t_x\sigma_x^2} - \frac{1}{t_x^2} - \frac{\{nx\}}{t_x} - \frac{\{nx\}\ell_3}{6\sigma_x^2} - \frac{\{nx\}^2}{2}\right)$$

where $\ell_3 = L^{(3)}(t_x)$ and $\ell_4 = L^{(4)}(t_x)$.

Recursive construction: We construct π_{n+1} from π_n as follows.



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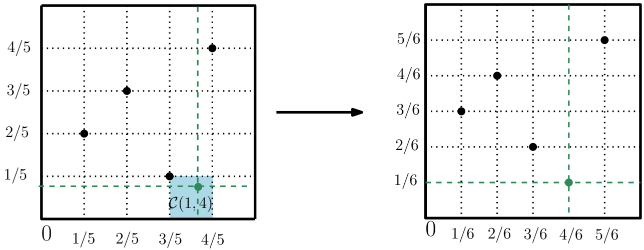


From this representation we can study the number of cells $\mathcal{C}(i,j)$ that generate prescribed increments $\Delta D_{n+1} = a$ and $\Delta D'_{n+1} = b$, for $a, b \in \{0,1\}$ in the permutation π_n and π_n^{-1} when $U_{n+1} \in \mathcal{C}(i,j)$.

Define for $a, b \in \{0, 1\}$, the number of cells with prescribed increments for the permutation and its inverse

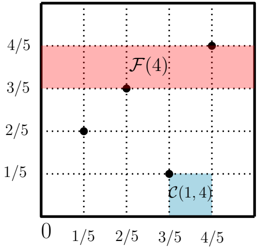
$$c_{a,b} := c_{a,b}(\pi_n) = |\{\mathcal{C}(i,j) : \Delta D_{n+1}(i,j) = a, \Delta D'_{n+1}(i,j) = b\}| \quad (1)$$

Theorem (BBFR '24)

Every permutation π_n, out of its $(n+1)^2$ cells, has exactly

$$\begin{aligned} c_{1,1} &= (n - D_n)(n - D'_n) + n \\ c_{1,0} &= (n - D_n)(D'_n + 1) - n \\ c_{0,1} &= (D_n + 1)(n - D'_n) - n \\ c_{0,0} &= (D_n + 1)(D'_n + 1) + n \end{aligned}$$

We define the *fiber of height* ℓ the set $\mathcal{F}(\ell) = \cup_{i=1}^{n+1} C(i, \ell)$.



Lemma (BBFR '24)

For each $\ell \in \{1, 2, \ldots, n+1\}$ the fiber $\mathcal{F}(\ell)$ has exactly

$$|\{i : \Delta D_{n+1}(i, \ell) = 1\}| = n - D_n \quad \text{and}$$
$$|\{i : \Delta D_{n+1}(i, \ell) = 0\}| = D_n + 1$$

Sketch of proof :

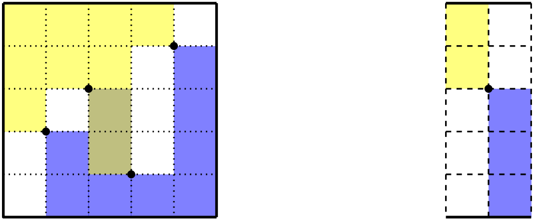


Figure: We color yellow the cells to the "up-left" of each point and blue the cells to the "low-right" of each point.

Properties:

1. Each point induces one yellow or a blue cell for each fiber.

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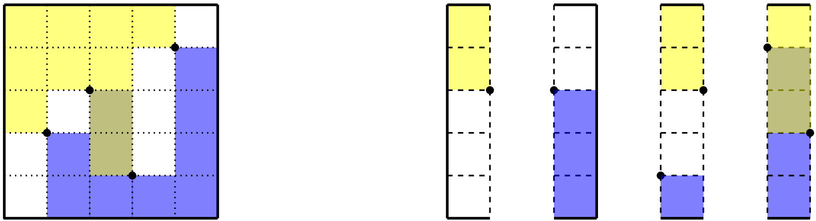

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Properties:

1. Each point induces one yellow or a blue cell for each fiber.
2. Reading vertical fibers there are four cases to have increment +1 :

   Left) if yellow

   Right) if blue

   Interior) two cases depending if there is descent on π_n:

   - Ascent) if exclusively yellow or blue.
   - Descent) if yellow and blue.

$$Y(1,\ell) + B(n+1,\ell) + \sum_{\substack{i\in\{2,\dots,n\}\\ \pi_n(i-1)<\pi_n(i)}} (Y(i,\ell)+B(i,\ell)) + \sum_{\substack{i\in\{2,\dots,n\}\\ \pi_n(i-1)>\pi_n(i)}} (Y(i,\ell)+B(i,\ell)-1)$$

$$= \sum_{i=1}^{n}(Y(i,\ell)+B(i+1,\ell)) - D_n$$

$$= n - D_n,$$

Corollary (BBFR '24)

For every π_n, out of the $(n+1)^2$ cells, there are exactly $(n+1)(n-D_n)$ with increment $+1$.

Sketch of proof of the Theorem

Claim: It is enough to prove that $c_{1,1} = (n - D_n)(n - D'_n) + n$.
Proof of claim: We recover $c_{0,0}$ from $c_{1,1}$.
From π_n we construct $\overline{\pi}_n(i) = n - \pi_n(i)$.
This exchanges descents with ascents for π_n and π_n^{-1}.
Ascents + Descents = $n - 1$.

$$c_{0,0}(\pi_n) = c_{1,1}(\bar{\pi}_n) = (D_n + 1)(D'_n + 1) + n.$$

For $c_{1,0}$ from $c_{1,1}$, thanks to the previous Corollary one has that

$$c_{1,0} + c_{1,1} = (n - D_n)(n + 1) \quad \Longrightarrow \quad c_{1,0} = (n - D_n)(D'_n + 1) - n.$$

Similarly for $c_{0,1}$.

Proof for $c_{1,1}$: Induction.
$n = 1$ identity, easy to check.
Inductive step : we consider π_{n+1} a permutation of size $n+1$ and we induce a permutation π_n of size n as follows

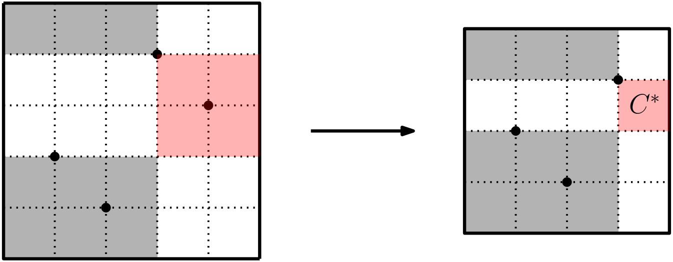

Figure: Permutation $\pi_n = (213)$ (to the right) induced for $n+1 = 4$ from $\pi_{n+1} = (2143)$ (to the left) when taking out $\pi_{n+1}(n+1) = 3$. We keep track of the increment behavior of the red cell during the induction.

The increments in the gray region remain invariant!

Depending on the behavior of $(\Delta D_{n+1}(C^*), \Delta D'_{n+1}(C^*))$ we have four cases.

We illustrate with one case : if it is equal to $(0, 1)$, we have

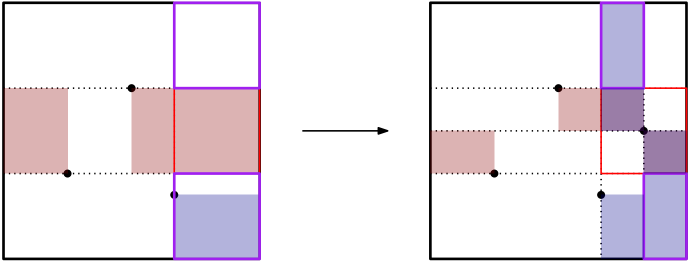

Figure: The blue cells are cells with $\Delta D_{n+1} = 1$ and the red cells are cells with $\Delta D'_{n+1} = 1$.

To conclude we apply to the fiber of π_n^{-1} the previous Lemma, i.e. rightmost fiber in the left image, has

$$|\{\mathcal{C}(n+1, j) : \Delta D'_{n+1}(n+1, j) = 1\}| = n - D'_n$$